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Extremal Extensions for m -jets from \mathbb{R} to \mathbb{R}^n .

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Abstract

We characterize the Lipschitz constant for the m -fields ($m \in \mathbb{N}$) from \mathbb{R} to \mathbb{R}^n . This work completes the results of J. Favard [10] and G. Glaeser [20] (see also [24]). Let us consider a m -field U . Our problem is to solve

$$\inf\{\text{Lip}(g^{(m)}) : g \text{ is an } m\text{-Lipschitz extension of } U\},$$

where Lip is the Lipschitz constant, and to characterize the *extremal* extension f , according to Favard's terminology [10], for which the above infimum is attained. The expression of the extremal extension contains the antiderivative of a rational function where the numerator is a polynomial and the denominator is the Euclidean norm of this polynomial. We further study the stability of this solution.

1 Introduction

We are interested in the problem initiated by H. Whitney [30], [31] concerning the extension of Taylorian fields. In this paper we consider the m -fields ($m \in \mathbb{N}$) from \mathbb{R} to \mathbb{R}^n . The main result of this paper generalized a result of G. Glaeser [21] which deals with the problem of *extremal* extension, according to Favard's terminology [10], for m -Taylorian fields from \mathbb{R} to \mathbb{R} .

Denote $\mathbb{R}_n[X]$ the space of polynomials from \mathbb{R} to \mathbb{R}^n . For any $m, n \in \mathbb{N}^*$, consider a m -field U

$$U: a \in \text{dom}(U) \rightarrow U_a \in \mathcal{P}_{m,n}$$

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where $\text{dom}(U)$ is a subset of \mathbb{R} and $\mathcal{P}_{m,n}$ (or \mathcal{P} for short) is the subspace of $\mathbb{R}_n[X]$ consisting of polynomials with real coefficients and degree at most m .

In the theory of fields, the first problem studied by Whitney [30], [31] and Glaeser [21] (also see [24]) is to find a necessary and sufficient condition for a field of polynomials of degree at most m , defined on a closed non empty subset of \mathbb{R}^d , to extend to (the field of the Taylorian m -expansions of) a m times continuously differentiable total function f on \mathbb{R}^d . We can therefore say that f is an extension of U or even that U is a m -Taylorian field.

Whitney's condition is given by the following theorem.

Theorem 1.1. (*Whitney, Glaeser*) *A necessary and sufficient condition for a m -field to extend to a m times continuously differentiable total function f is that there exists a modulus of continuity ω such that for any $a, b \in \text{dom}(U)$, for any $k \in \{0, \dots, m\}$*

$$\frac{\|U_b^{(m-k)}(b) - U_a^{(m-k)}(b)\|}{|b - a|^k} \leq \omega(|b - a|). \quad (1.1)$$

When $\omega(h) = Ch$, where C is a positive constant, then there are extensions f with $f^{(m)}$ m -Lipschitzian. In that case, we say that f is a m -Lipchitzian extension of U .

Define $\mathcal{L}(U)$ by

$$\mathcal{L}(U) := \inf\{\text{Lip}(g^{(m)}) : g \text{ is an } m\text{-Lipschitz extension of } U\}.$$

We say that an extension f of U is a minimal Lipschitz extension if

$$\text{Lip}(f^{(m)}) = \mathcal{L}(U). \quad (1.2)$$

In its weakest form, this problem can be stated as follows:

Does there exist a total differentiable function which is a minimal Lipschitz extension of U ? Restricting to a compact of \mathbb{R} , this problem has a positive solution which follows from Glaeser's extension theorem [20] and an Ascoli's type theorem for fields of jets [24]. In its strong form the problem which we study in this article is, on the one hand to characterize $\mathcal{L}(U)$ uniquely in function of the values of U , and on the other hand to have an explicit expression of an extension f which satisfies (1.2). In general there is not uniqueness of the minimal Lipschitz extension except then U is a biponctual m -field (see Theorem 3.4). This consideration allows us to define an extremal Lipschitz extension.

For the multivariate continuous minimal Lipschitz extension problem and for $n = 1$, the earliest result in this direction is the result of Mac Shane [27] and Whitney [[30], footnote p. 63] see also [2], [26] and [25], and for any n the result of M.D. Kirszbraun [23], see also the textbook [[11], p. 199] and the generalization [29], [9].

In [21], Glaeser characterized the value of $\mathcal{L}(U)$ for univariate m -fields and for $n = 1$, and proved that the solution to the problem is a perfect spline, that is to say a piecewise polynomial function of degree at most $m + 1$ with \mathbb{R} as their natural domain of definition.

In [24], for fields defined in a Hilbert space and for $n = 1$ we have characterized the best Lipschitz constant.

In this article we also answer this question for the univariate fields and for any m and n . In this case the characterization of $\mathcal{L}(U)$ is similar to that introduced in [21] and the mathematical techniques used for the proof of the result are close to those used in the cited article. We show the uniqueness of the solution of bipunctual fields. Contrarily to the case solved in [21], the solution is not always a piecewise polynomial function but its expression may contain the antiderivative of a rational function where the numerator is a polynomial in \mathcal{P} and the denominator is the Euclidean norm of this polynomial.

In our opinion, the mathematical techniques used here cannot be generalized to the Whitney's extension problem for multivariable \mathcal{C}^m -functions see [6],[7],[[12],... , [19]], [28], and [32] as, in that case, we do not have uniqueness of a minimal extension for a bipunctual field see [24], which holds true for univariate fields and which is fundamental in the proof.

This paper is organized as follows. In the first section, we introduce the notations and definitions. We also provide some elementary properties. The second section is devoted to the study of bipunctual m -fields. The main theorem 3.4 proves the existence and uniqueness of an m -Lipschitzian extension satisfying (1.2). We characterize $\mathcal{L}(U)$ and give the expression of the solution. The geometrical Lemma 3.1, used in the proof of theorem 3.4, proves the uniqueness of a supporting hyperplane containing a point of the unit ball.

In the third section we study the extremal extension for any m -field. We characterize the Lipschitz constant for a m -field (see theorem 3.7). A Whitney type theorem (see 3.9) is established as well as the stability properties of the solution (see 3.10).

The last section is an annex containing the different Lemmas used in this paper see 4.1, 4.2, and 4.3.

2 Notations and definitions

In what follows we denote by m, n two strictly positive integers, by \mathcal{P}_m the set of univariate polynomials of degree at most m .

We set $\mathcal{P} := (\mathcal{P}_m)^n$.

For $P \in \mathcal{P}$, we denote by (p_1, \dots, p_n) the coordinates of P in the standard basis denoted by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$.

For $i \in \{1, \dots, n\}$, $t \in \mathbb{R}$, we denote $p_i(t) := \sum_{j=0}^m p_{i,j} t^j$.

We set $d := n(m+1)$ and $\mathcal{A} := (O; \mathbb{R}^d)$ the d -dimensional affine space.

For $P \in \mathcal{P}$, we set $\Lambda(P) := (p_{i,j})_{i=1,\dots,n;j=0,\dots,m} \in \mathcal{A}$.

The usual inner product defined on \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm by $\| \cdot \|$.

Definition 2.1. A m -field U is defined by

$$U: a \in \text{dom}(U) \subset \mathbb{R} \rightarrow U_a \in \mathcal{P}.$$

Definition 2.2. Let U be a m -field.

We say that the function f from \mathbb{R} to \mathbb{R}^n is a m -Lipschitz extension of U if and only if

- f is a m -times differentiable total function on \mathbb{R} ,
- $f^{(m)}$ is a Lipschitz function,
- the m -Taylorian expansion of f coincides with U in restriction to $\text{dom}(U)$.

Let U be a m -field. Denote by $\mathcal{E}(U)$ the set of m -Lipschitz extensions of U .

For $a \in \text{dom}(U)$ and $x \in \mathbb{R}$, we denote:

$$U_a(x) := \sum_{k=0}^m \frac{(x-a)^k}{k!} U_a^{(k)}$$

with $U_a^{(k)} := (U_{a,1}^{(k)}, \dots, U_{a,n}^{(k)})$.

For $a, b \in \text{dom}(U)$, $x \in \mathbb{R}$ and $\phi \in \mathcal{E}(U)$ we have the following Taylorian formula:

$$U_b(x) - U_a(x) = \int_a^b \frac{(x-t)^m}{m!} \phi^{(m+1)}(t) dt. \quad (2.1)$$

(see Annex 4.2)

Definition 2.3. For any pair $(P, Q) \in \mathcal{P} \times \mathcal{P}$ we define:

$$\ll P; Q \gg := \sum_{i=1}^n \sum_{k=0}^m ((-1)^{m-k} p_i^{(k)}(b) q_i^{(m-k)}(b)). \quad (2.2)$$

We have the following Lemma

Lemma 2.4. *Let U be a bimonotone m -field. Let $\psi \in \mathcal{E}(U)$. Then for any $P \in \mathcal{P}$ we have*

$$\ll U_b - U_a; P \gg = \int_a^b \langle P(t); \psi^{(m+1)}(t) \rangle dt. \quad (2.3)$$

For the proof of this Lemma see annex 4.1.

The unit ball \mathcal{B} that we consider here is defined as

$$\mathcal{B} := \{\Lambda(P) \in \mathcal{A} : \int_a^b \|P(t)\| dt = b - a\}.$$

Let H be a hyperplane of the affine space \mathcal{A} , and $a, b \in \mathbb{R}$, with $a < b$ then from Lemma 4.1 (see annex), there exists $\phi \in \mathbb{L}^\infty([a, b]; \mathbb{R}^n)$ such that

$$H = H_\phi := \{\Lambda(P) : \int_a^b \langle P(t); \phi(t) \rangle dt = b - a\}. \quad (2.4)$$

In other words, H can be represented in the form (2.4).

Definition 2.5. Let $\Lambda(P) \in \mathcal{B}$. The hyperplane H_ϕ is a supporting hyperplane to the unit ball \mathcal{B} containing $\Lambda(L)$ if and only if

$$(\mathbf{a}_1) \quad \Lambda(L) \in H_\phi, \quad (\mathbf{a}_2) \quad \forall \Lambda(P) \in H_\phi, \quad \int_a^b \|P(t)\| dt \geq b - a.$$

For $\phi \in \mathbb{L}^\infty([a, b]; \mathbb{R}^n)$, and $a \neq b$ we denote

$$|\phi|_{\infty, [a, b]} := \sup_{t \in [a, b]} \|\phi(t)\|.$$

3 Resolution of the extremal extension problem for bimonotone m -field

In this section, n et m are fixed.

3.1 Uniqueness of a supporting hyperplane

Lemma 3.1. *Let $\Lambda(L) \in \mathcal{B}$.*

There exists a unique supporting hyperplane to the unit ball \mathcal{B} containing $\Lambda(L)$. Furthermore this hyperplane can be represented in the form

$$H := \{\Lambda(P) : \int_a^b \langle P(t); \frac{L(t)}{\|L(t)\|} \rangle dt = b - a\}.$$

Proof. Existence.

For the existence, it is easy to verify that H is a solution. Indeed

$$\int_a^b \langle L(t); \frac{L(t)}{\|L(t)\|} \rangle dt = \int_a^b \|L(t)\| dt = b - a.$$

Thus $\Lambda(L) \in H$. Let $\Lambda(P) \in H$, we have

$$b - a = \int_a^b \langle P(t); \frac{L(t)}{\|L(t)\|} \rangle dt \leq \int_a^b \|P(t)\| dt.$$

Therefore H satisfies the properties (\mathbf{a}_1) and (\mathbf{a}_2) .

Uniqueness. Consider another supporting hyperplane to the unit ball \mathcal{B} containing $\Lambda(L)$ which can be represented in the form

$$H_\phi := \{\Lambda(P) : \int_a^b \langle P(t); \phi(t) \rangle dt = b - a\},$$

where $\phi = \frac{L}{\|L\|} + \delta$, with $\delta \in \mathbb{L}^\infty([a, b]; \mathbb{R}^n)$ (see Lemma 4.1).

We notice that if

$$\int_a^b \langle P(t); \delta(t) \rangle dt = 0, \forall \Lambda(P) \in H_\phi, \quad (3.1)$$

then $H_\phi \subset H_\psi$ and consequently $H_\phi = H_\psi$.

Thus to prove the uniqueness it is sufficient to prove (3.1).

Since $\Lambda(L) \in H_\phi$, we have :

$$\int_a^b \langle L(t); \delta(t) \rangle dt = 0$$

Thus it is sufficient to prove

$$\int_a^b \langle Q(t); \delta(t) \rangle dt = 0, \forall \Lambda(P) = \Lambda(L + Q) \in H_\phi. \quad (3.2)$$

Let $\Lambda(P) = \Lambda(L + Q) \in H_\phi$, we have:

$$\int_a^b \langle Q(t); \frac{L(t)}{\|L(t)\|} + \delta(t) \rangle dt = 0. \quad (3.3)$$

Since H_ϕ is a supporting hyperplane we have

$$\int_a^b \|L(t) + Q(t)\| dt \geq b - a = \int_a^b \|L(t)\| dt. \quad (3.4)$$

We deduce the following inequality

$$\int_a^b \langle Q(t); \delta(t) \rangle dt \leq \int_a^b \Delta(t) dt, \quad (3.5)$$

where

$$\Delta(t) := \|L(t) + Q(t)\| - \|L(t)\| - \langle Q(t); \frac{L(t)}{\|L(t)\|} \rangle.$$

For the rest, we notice that if $\Lambda(L + Q) \in H_\phi$ then $\Lambda(L + \alpha Q) \in H_\phi$, for every $\alpha \in \mathbb{R}$.

Let $t \in [a, b]$. We have $\Delta(t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t)$ with :

$$\begin{aligned} \Delta_1(t) &= \frac{\|Q(t)\|^2}{\|L(t) + Q(t)\| + \|L(t)\|}, \\ \Delta_2(t) &= -\frac{\|Q(t)\|^2}{(\|L(t) + Q(t)\| + \|L(t)\|)^2} \langle Q(t); \frac{L(t)}{\|L(t)\|} \rangle, \\ \Delta_3(t) &= -2\frac{\|L(t)\|}{(\|L(t) + Q(t)\| + \|L(t)\|)^2} (\langle Q(t); \frac{L(t)}{\|L(t)\|} \rangle)^2. \end{aligned}$$

By replacing Q by αQ , for $\alpha > 0$ in (3.5) and by noticing that $\Delta_3(t) \leq 0$, inequality (3.5) implies the following inequality:

$$\int_a^b \langle Q(t); \delta(t) \rangle dt \leq \alpha(\Theta_1(\alpha) + \alpha\Theta_2(\alpha)), \quad (3.6)$$

where

$$\Theta_1(\alpha) := \int_a^b \frac{\|Q(t)\|^2}{\|L(t) + \alpha Q(t)\| + \|L(t)\|} dt,$$

and

$$\Theta_2(\alpha) := -\int_a^b \frac{\|Q(t)\|^2}{(\|L(t) + \alpha Q(t)\| + \|L(t)\|)^2} \langle Q(t); \frac{L(t)}{\|L(t)\|} \rangle dt.$$

We notice that if $r \in [a, b]$ satisfies: $\|L(r) + \alpha Q(r)\| + \|L(r)\| = 0$ then r is a root of the polynomials l_i and q_i for $i \in \{1, \dots, n\}$.

We infer that there exists $\tilde{L}, \tilde{Q} \in \mathcal{P}$ which satisfies:

$$\|\tilde{L}(t) + \alpha \tilde{Q}(t)\| \neq 0, \|\tilde{L}(t)\| \neq 0, \forall t \in [a, b],$$

and

$$\frac{\|\tilde{Q}(t)\|}{\|\tilde{L}(t) + \alpha \tilde{Q}(t)\| + \|\tilde{L}(t)\|} = \frac{\|Q(t)\|}{\|L(t) + \alpha Q(t)\| + \|L(t)\|}.$$

This enables us to obtain the following uniform majorations:

$$\Theta_1(\alpha) \leq \int_a^b \frac{\|\tilde{Q}(t)\| \|Q(t)\|}{\|\tilde{L}(t)\|} dt < \infty;$$

and

$$\Theta_2(\alpha) \leq \int_a^b \frac{\|\tilde{Q}(t)\|^2 \|Q(t)\|}{\|\tilde{L}(t)\|^2} dt < \infty.$$

The limit when α tends to 0 in the inequality (3.6) gives

$$\int_a^b \langle Q(t); \delta(t) \rangle dt \leq 0.$$

Since $L - Q \in H_\phi$ we also have:

$$\int_a^b \langle -Q(t); \delta(t) \rangle dt \leq 0,$$

and this allows us to obtain the required following equality

$$\int_a^b \langle Q(t); \delta(t) \rangle dt = 0.$$

We conclude that $H_\phi = H$.

Remark 3.2. We still have $\|\phi\|_{\infty, [a, b]} \geq 1$. If we assume that $\|\phi\|_{\infty, [a, b]} = 1$ then we have another nice proof of uniqueness, see annex 4.2. We have not found the simple argument, if it exists, allowing us to state this supplementary hypothesis.

Remark 3.3. There may be several $\Lambda(L) \in \mathcal{B}$ representing the same supporting hyperplane.

□

3.2 Extremal extension for a bipoctual m -field

We will characterize the unique extremal extension as well as the Lipschitz constant of a bipoctual m -field.

Let $U = \{U_a, U_b\}$, with $a < b$, be a bipoctual m -field. Recall that

$$\mathcal{L}(U) := \inf\{\text{Lip}(g^{(m)}; [a, b]) : g \in \mathcal{E}(U)\}. \quad (3.7)$$

Theorem 3.4. *If $U = \{U_a, U_b\}$, $a < b$ is a bipoctual m -field. Then*

- (1) *There exists a unique extension $f \in \mathcal{E}(U)$ by restricting to the segment $[a, b]$ such that*

$$\text{Lip}(f^{(m)}; [a, b]) = \mathcal{L}(U). \quad (3.8)$$

- (2) *We define*

$$\mathcal{K}(U) := \sup_{\Lambda(P) \in \mathcal{B}} \ll P; U_b - U_a \gg. \quad (3.9)$$

Then

$$\mathcal{L}(U) = \frac{\mathcal{K}(U)}{b - a}. \quad (3.10)$$

- (3) *Let $\Lambda(L) \in \mathcal{B}$ such that $\ll L; U_b - U_a \gg = \mathcal{K}(U)$. Then*

$$f(x) = U_a(x) + \mathcal{L}(U) \int_a^x \frac{(x-t)^m}{m!} \frac{L(t)}{\|L(t)\|} dt. \quad (3.11)$$

The equality (3.10) shows that $\frac{\mathcal{K}(U)}{b-a}$ is the inner expression of $\mathcal{L}(U)$, and the equality (3.11) characterizes the unique extremal extension for a bipoctual m -field.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Let us consider a bipoctual m -jet, $U := \{U_a, U_b\}$.

From Lemma 2.4, we have for any $g \in \mathcal{E}(U)$ and for any $P \in \mathcal{P}$

$$\ll P; U_b - U_a \gg = \int_a^b \langle P(t); g^{(m+1)}(t) \rangle dt.$$

We infer the following majoration for all $g \in \mathcal{E}(U)$, and $P \in \mathcal{P}$

$$\ll P; U_b - U_a \gg \leq \|g^{(m+1)}\|_{\infty, [a, b]} \int_a^b \|P(t)\| dt. \quad (3.12)$$

We set $\mathcal{K}(U) := \sup_{\Lambda(P) \in \mathcal{B}} \ll P; U_b - U_a \gg$.

By the last inequality, we obtain

$$\forall g \in \mathcal{E}(U), \quad \mathcal{K}(U) \leq (b - a) \|g^{(m+1)}\|_{\infty, [a, b]}. \quad (3.13)$$

The unit ball \mathcal{B} being compact, there exists $\Lambda(L) \in \mathcal{B}$ such that

$$\ll L; U_b - U_a \gg = \mathcal{K}(U). \quad (3.14)$$

Denote by L the polynomial which satisfies the last equality, and consider the following hyperplane:

$$H := \{\Lambda(P) \in \mathcal{A} : \ll P; U_b - U_a \gg = \mathcal{K}(U)\}.$$

Obviously $\Lambda(L) \in H$, and for $\Lambda(P) \in H$ we have:

$$\mathcal{K}(U) \frac{(b - a)}{\int_a^b \|P(t)\| dt} = \frac{(b - a)}{\int_a^b \|P(t)\| dt} \ll P; U_b - U_a \gg \leq \mathcal{K}(U).$$

We infer that $\int_a^b \|P(t)\| dt \geq b - a$. Therefore H is a supporting hyperplane to the unit ball \mathcal{B} containing $\Lambda(L)$.

From the geometric Lemma 3.1, there exists a unique supporting hyperplane containing $\Lambda(L)$, and furthermore it can be represented in the form

$$H := \{\Lambda(P) \in \mathcal{A} : \int_a^b \langle P(t); \frac{L(t)}{\|L(t)\|} \rangle dt = b - a\}.$$

We infer the following equality:

$$\forall \Lambda(P) \in H, \quad \ll P; U_b - U_a \gg = \frac{\mathcal{K}(U)}{b - a} \int_a^b \langle P(t); \frac{L(t)}{\|L(t)\|} \rangle dt. \quad (3.15)$$

Now we will check that this equality remains true for all $Q \in \mathcal{P}$.
Let $Q \in \mathcal{P}$. Write Q as $Q = P + \alpha L$ with

$$\alpha = -1 + \frac{1}{b - a} \int_a^b \langle Q(t); \frac{L(t)}{\|L(t)\|} \rangle dt.$$

By noting that if $\alpha = 0$ then $\Lambda(Q) \in H$, we assume that $\alpha \neq 0$. An elementary calculation shows that $\Lambda(P)$ belongs to H .

Therefore the polynomial $Q - \alpha L$ satisfies the equality

$$\ll Q - \alpha L; U_b - U_a \gg = \frac{\mathcal{K}(U)}{b-a} \int_a^b \langle Q(t) - \alpha L(t); \frac{L(t)}{\|L(t)\|} \rangle dt.$$

Thus

$$\ll L; U_b - U_a \gg = \frac{\mathcal{K}(U)}{b-a} \int_a^b \|L(t)\| dt.$$

We infer that Q satisfies the equality (3.15). We can also note that this equality is satisfied for any Q due to the fact that the hyperplane H does not contain the origin of the affine space \mathcal{A} .

Hence

$$\forall \Lambda(P) \in \mathcal{A}, \quad \ll P; U_b - U_a \gg = \frac{\mathcal{K}(U)}{b-a} \int_a^b \langle P(t); \frac{L(t)}{\|L(t)\|} \rangle dt. \quad (3.16)$$

Here is now the solution to the problem.

We define the function $f: [a, b] \rightarrow \mathbb{R}^n$ by setting for $x \in [a, b]$

$$f(x) = U_a(x) + \frac{\mathcal{K}(U)}{b-a} \int_a^x \frac{(x-t)^m}{m!} \frac{L(t)}{\|L(t)\|} dt. \quad (3.17)$$

By using (3.16) we will verify that $f \in \mathcal{E}(U)$ on restriction to $[a, b]$.

If $k \in \{1, \dots, n\}$, we have

$$f^{(k)}(x) = U_a^{(k)}(x) + \frac{\mathcal{K}(U)}{b-a} \int_a^x \frac{(x-t)^{m-k}}{(m-k)!} \frac{L(t)}{\|L(t)\|} dt.$$

- If $x = a$ we have $f^{(k)}(a) = U_a^{(k)}(a)$. Therefore the m -Taylorian expansion of f at the point a coincides with U_a .

- If $x = b$ we have

$$f^{(k)}(b) = U_a^{(k)}(b) + \frac{\mathcal{K}(U)}{b-a} \int_a^b \frac{(b-t)^{m-k}}{(m-k)!} \frac{L(t)}{\|L(t)\|} dt. \quad (3.18)$$

For $i \in \{1, \dots, n\}$, we use the equality (3.16) for $P(t) = q(t)\mathbf{e}_i$ with

$$q(t) := \frac{(b-t)^{m-k}}{(m-k)!}.$$

From the definition 2.2 from $\ll \cdot; \cdot \gg$, we have :

$$\begin{aligned} \ll P; U_b - U_a \gg &= \sum_{j=1}^n \sum_{l=0}^m (-1)^l p_j^{(l)}(b) (U_{b,j}^{(m-l)}(b) - U_{a,j}^{(m-l)}(b)) \\ &= \sum_{l=0}^m (-1)^l p_i^{(l)}(b) (U_{b,i}^{(m-l)}(b) - U_{a,i}^{(m-l)}(a)) \\ &= (-1)^{m-k} (-1)^{m-k} (U_{b,i}^k(b) - U_{a,i}^k(b)) \\ &= U_{b,i}^k(b) - U_{a,i}^k(b). \end{aligned}$$

The equality (3.16) applied to P gives

$$U_{b,i}^k(b) - U_{a,i}^k(b) = \frac{\mathcal{K}(U)}{b-a} \int_a^b \frac{(b-t)^{m-k}}{(m-k)!} \frac{L_i(t)}{\|L(t)\|} dt.$$

Using (3.18) we have :

$$f_i^{(k)}(b) = U_{a,i}^{(k)}(b) + U_{b,i}^k(b) - U_{a,i}^k(b) = U_{b,i}^k(b).$$

Therefore the m -Taylorian expansion of f at the point b coincides with U_b .

- If $k = m + 1$, we obtain

$$f^{(m+1)}(x) = \frac{\mathcal{K}(U)}{b-a} \frac{L(x)}{\|L(x)\|}.$$

We infer the equality

$$\|f^{(m+1)}\|_{\infty, [a, b]} = \frac{\mathcal{K}(U)}{b-a}.$$

In conclusion, we obtain, on the one hand, that $f \in \mathcal{E}(U)$ and on the other hand, from (3.13) that

$$\forall g \in \mathcal{E}(U) : \text{Lip}(f^{(m)}; [a, b]) \leq \text{Lip}(g^{(m)}; [a, b])$$

The function f is therefore the unique extremal extension of m -field U and $\frac{\mathcal{K}(U)}{b-a}$ is an inner expression of $\mathcal{L}(U)$, as desired. \square

Remark 3.5. The uniqueness of f comes from the fact that there is uniqueness of the supporting hyperplane H .

3.3 Extremal extensions for the m -fields

As in [24] we will define the Lipschitz constant of a m -Lipschitz field.

Definition 3.6. Let U be a m -field. For any $a, b \in \text{dom}(U)$, $a \neq b$, we set

$$\gamma_{a,b}(U) := \frac{K_{a,b}}{|b-a|} := \sup_{P \in \mathcal{P} \setminus \{0\}} \frac{\ll P; U_b - U_a \gg}{\int_a^b \|P(t)\| dt},$$

and

$$\Gamma(U) := \sup_{a \neq b \in \text{dom}(U)} \gamma_{a,b}(U).$$

Theorem 3.7. $\Gamma: U \mapsto \Gamma(U)$ is the unique operator which satisfies

- (\mathbf{P}_0) For all m -fields U and V such that $\text{dom}(U) \subset \text{dom}(V)$, and V extends U we have

$$\Gamma(U) \leq \Gamma(V).$$

- (\mathbf{P}_1) For every total function f in $\mathcal{C}^m(\mathbb{R}; \mathbb{R}^n)$, such that $\text{Lip}(f^{(m)}) < \infty$ we have

$$\Gamma(F) = \text{Lip}(f^{(m)}).$$

where F is a m -Taylorian expansion of f .

- (\mathbf{P}_2) Let F be a total m -field such that $\Gamma(F) < \infty$. Let us consider $f(x) := F_x(x)$ for every $x \in \mathbb{R}$. Then $f \in \mathcal{C}^m(\mathbb{R}; \mathbb{R}^n)$, $\text{Lip}(f^{(m)}) = \Gamma(F)$ and the m -Taylorian expansion of f coincides with F .

- (\mathbf{P}_3) For any m -field U , with $\Gamma(U) < \infty$, there exists a total m -field F such that

$$\Gamma(F) = \Gamma(U).$$

In other words, let U be a m -field, with $\mathcal{L}(U) < \infty$. Denote by A the domain of U . We have

$$\Gamma(U) = \mathcal{L}(U).$$

Thus $\Gamma(U)$ is the inner expression of $\mathcal{L}(U)$.

In addition, for any pair $(a, b) \in A \times A$ such that $a < b$, $]a, b[\cap A = \emptyset$, and $x \in [a, b]$ we denote by f the unique extremal extension of the biponctual m -field $\{U_a, U_b\}$. Recall that f is defined by the formula

$$f(x) := U_a(x) + \frac{K_{a,b}}{b-a} \int_a^x \frac{(x-t)^m}{m!} \frac{L_{a,b}(t)}{\|L_{a,b}(t)\|} dt,$$

where $K_{a,b}$ and $L_{a,b}$ are obtained by formulas (3.9) and (3.11) see Theorem 3.4. Let us consider the following function

$$\tilde{f}(x) := \begin{cases} U_x(x) & \text{if } x \in A \\ f(x) & \text{if } x \notin A. \end{cases}$$

Then \tilde{f} is the unique extremal extension of U .

Remark 3.8. The property (\mathbf{P}_3) is crucial as it means that $\Gamma(U)$ is the *inner* expression of $\mathcal{L}(U)$.

Proof. The property (\mathbf{P}_0) is easy to verify. Let us prove (\mathbf{P}_1) . Let f be a total function in \mathcal{C}^m with $\text{Lip}(f^{(m)}) < \infty$. Denote by F the m -Taylorian expansion of f .

Let us consider $P \in \mathcal{P} \setminus \{0\}$, and $a, b \in \mathbb{R}$. Since f is an extension of the biponctual m -field $\{F_a, F_b\}$ we have the equality

$$\ll P; F_b - F_a \gg = \langle P(b); f^{(m)}(b) \rangle - \langle P(a); f^{(m)}(a) \rangle - \int_a^b \langle P'(t); f^{(m)}(t) \rangle dt.$$

Since the function $f^{(m)}$ is Lipschitzian, we have the equality

$$\langle P(b); f^{(m)}(b) \rangle - \langle P(a); f^{(m)}(a) \rangle - \int_a^b \langle P(t); f^{(m)}(t) \rangle dt = \int_a^b \langle P(t); f^{(m+1)}(t) \rangle dt$$

and the majoration

$$\int_a^b \langle P(t); f^{(m+1)}(t) \rangle dt \leq \sup_{t \in [a,b]} \|f^{(m+1)}\| \int_a^b \|P(t)\| dt \leq \text{Lip}(f^{(m)}) \int_a^b \|P(t)\| dt.$$

Therefore

$$\Gamma(F) \leq \text{Lip}(f^{(m)}).$$

Let us consider

$$Q := \sum_{i=1}^n (F_{b,i}^{(m)}(b) - F_{a,i}^{(m)}(a)) \mathbf{e}_i.$$

Since

$$\ll Q; F_b - F_a \gg = \|F_b^{(m)} - F_a^{(m)}\|^2,$$

and

$$\int_a^b \|P(t)\| dt = |b - a| \|F_b^{(m)} - F_a^{(m)}\|,$$

we obtain the equality

$$\frac{\ll Q; U_b - U_a \gg}{\int_a^b \|Q(t)\| dt} = \frac{\|F_b^{(m)} - F_a^{(m)}\|}{|b - a|} \leq \Gamma(F).$$

Therefore

$$\text{Lip}(f^{(m)}) \leq \Gamma(F),$$

and (\mathbf{P}_1) is proved.

Let us prove (\mathbf{P}_2) . Let U be a total m -field with $\Gamma(U) < \infty$. Setting $u(x) := U_x(x)$ for every $x \in \mathbb{R}$, and

for $k \in \{0, \dots, m\}$

$$Q_k := \sum_{i=1}^n (U_{b,i}^{(m-k)}(b) - U_{a,i}^{(m-k)}(b)) \frac{(b-t)^k}{k!} \mathbf{e}_i.$$

We have

$$\ll Q_k; U_b - U_a \gg = \|U_b^{(m-k)} - U_a^{(m-k)}\|^2,$$

and

$$\int_a^b \|Q_k(t)\| dt = \frac{|b-a|^{k+1}}{(k+1)!} \|U_b^{(m-k)} - U_a^{(m-k)}\|. \quad (3.19)$$

These equalities imply

$$\frac{\|U_b^{(m-k)}(b) - U_a^{(m-k)}(b)\|}{|b-a|^k} \leq \frac{|b-a|}{(k+1)!} \Gamma(U).$$

Therefore the m -field U satisfies Whitney's condition (1.1) for $\omega(h) = h\Gamma(U)$. Thus the function u is in \mathcal{C}^m , and this m -Taylorian expansion coincides with U on the total domain. The inequality (3.19) for $k = 0$ shows that

$$\text{Lip}(u^{(m)}) \leq \Gamma(U).$$

Applying now the property (\mathbf{P}_1) at u , we obtain the inverse inequality, therefore

$$\text{Lip}(u^{(m)}) = \Gamma(U),$$

and (\mathbf{P}_2) is proved.

Let us prove the last property (\mathbf{P}_3) . Let U be a m -Taylorian field such that $\Gamma(U) < \infty$. Setting $A := \text{dom}(U)$.

If A is not a closed set, since U satisfies Whitney's condition (3.9), by theorem [[24] p. 242] we can extend the field U to the closure of A with the same modulus of continuity. Therefore we consider now a closed set A .

For any pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, $]a, b[\cap A = \emptyset$, and $x \in [a, b]$, let us set

$$f(x) := U_a(x) + \frac{K_{a,b}}{b-a} \int_a^x \frac{(x-t)^m}{m!} \frac{L_{a,b}(t)}{\|L_{a,b}(t)\|} dt,$$

where $K_{a,b}$ and $L_{a,b}$ are obtained by formulas (3.9) and (3.11) see Theorem 3.4 for the biponctual m -field $\{U_a, U_b\}$.

Let us consider the m -field F which is the m -Taylorian expansion of f at x if $x \in \mathbb{K} \setminus A$, and $F_x := U_x$ if $x \in A$.

To prove (\mathbf{P}_3) it is sufficient to prove that for every $x, y \in \mathbb{K}$, $x \neq y$

$$\frac{K_{x,y}}{|y-x|} := \sup_{P \in \mathcal{P} \setminus \{0\}} \frac{\ll P; F_y - F_x \gg}{\int_x^y \|P(t)\| dt} \leq \Gamma(U).$$

We consider two cases.

First case. There exist $a, b \in A$ such that $a \leq x < y \leq b$, and $]a, b[\cap A = \emptyset$.

Since f is an extension of $\{F_x, F_y\}$ we can apply inequality (3.16). We have for every $P \in \mathcal{P}$

$$\ll P; F_y - F_x \gg \leq \frac{K_{a,b}}{b-a} \int_x^y \langle P(t); \frac{L_{a,b}(t)}{\|L_{a,b}(t)\|} \rangle dt.$$

Therefore

$$\frac{K_{x,y}}{|y-x|} \leq \frac{K_{a,b}}{|b-a|}.$$

Using theorem 3.4, we have uniqueness of the minimal extension of the m -field $\{U_a, U_b\}$ on $[a, b]$ thus

$$\frac{K_{x,y}}{|y-x|} = \frac{K_{a,b}}{|b-a|} \leq \Gamma(U).$$

Second case. There exist $a, b, c, d \in A$ such that $a \leq x < b \leq c \leq y \leq d$, with $]a, b[\cap A = \emptyset$, and $]c, d[\cap A = \emptyset$.

Let $P \in \mathcal{P}$ we have

$$\ll P; F_y - F_x \gg = \ll P; F_y - U_c \gg + \ll P; U_c - U_b \gg + \ll P; U_b - F_x \gg.$$

Setting $\Delta = \int_x^y \|P(t)\| dt$, $\Delta_1 = \int_c^y \|P(t)\| dt$, $\Delta_2 = \int_b^c \|P(t)\| dt$, and $\Delta_3 = \int_x^b \|P(t)\| dt$.

Since

$$\frac{\ll P; F_y - U_c \gg}{\Delta_1} \leq \frac{K_{c,y}}{y-c} = \frac{K_{c,d}}{d-c},$$

$$\frac{\ll P; U_b - U_c \gg}{\Delta_1} \leq \frac{K_{b,c}}{c-b},$$

$$\frac{\ll P; U_c - F_x \gg}{\Delta_1} \leq \frac{K_{x,b}}{b-x} = \frac{K_{a,b}}{b-a},$$

we obtain the majoration

$$\frac{\ll P; F_y - F_x \gg}{\Delta} \leq \frac{\Delta_1}{\Delta} \frac{K_{c,d}}{d-c} + \frac{\Delta_2}{\Delta} \frac{K_{b,c}}{c-b} + \frac{\Delta_3}{\Delta} \frac{K_{a,b}}{b-a}.$$

for every $P \in \mathcal{P} \setminus \{0\}$. Thus

$$\frac{K_{x,y}}{y-x} \leq \frac{\Delta_1}{\Delta} \frac{K_{c,d}}{d-c} + \frac{\Delta_2}{\Delta} \frac{K_{b,c}}{c-b} + \frac{\Delta_3}{\Delta} \frac{K_{a,b}}{b-a} \leq \Gamma(U).$$

In conclusion

$$\Gamma(F) = \Gamma(U),$$

and property (\mathbf{P}_3) is proved. \square

The following theorem gives a condition of Whitney's type.

Theorem 3.9. *Let U be a m -jet. The m -field U can be extended if and only if there exists a concave modulus of continuity ω such that*

$$\forall a \neq b \in \text{dom}(U), \quad K_{a,b} \leq \omega(|b-a|), \quad (3.20)$$

with

$$\frac{K_{a,b}}{|b-a|} = \sup_{P \in \mathcal{P} \setminus \{0\}} \frac{\ll P; U_b - U_a \gg}{\int_a^b \|P(t)\| dt}.$$

Proof. Let us consider a m -field U .

First suppose that there exists a concave modulus of continuity ω which satisfies (3.20). Then for any $a \neq b \in \text{dom}(U)$, for $k \in \{0, \dots, m\}$, and

$$Q_k := \sum_{i=1}^n (U_{b,i}^{(m-k)}(b) - U_{a,i}^{(m-k)}(b)) \frac{(b-t)^k}{k!} \mathbf{e}_i.$$

we have

$$\frac{\|U_b^{(m-k)}(b) - U_a^{(m-k)}(b)\|}{|b-a|^{k+1}} \leq \frac{\ll Q_k; U_b - U_a \gg}{\int_a^b \|Q_k(t)\| dt} \leq \frac{K_{a,b}}{|b-a|} \leq \frac{\omega(|b-a|)}{|b-a|}.$$

Thus U satisfies Whitney's condition (1.1) and it can be extended, as desired.

To prove the reverse implication, suppose that U satisfies Whitney's condition (1.1), and denote by ω the associated concave modulus of continuity.

Let $P = (p_1, \dots, p_m) \in \mathcal{P} \setminus \{0\}$. Let us set

$$\Delta_i := \sum_{k=0}^m (-1)^k p_i^{(k)}(b) (U_{b,i}^{(m-k)}(b) - U_{a,i}^{(m-k)}(b)).$$

Recall the following result of Glaeser (see Lemma 1 [[21], p. 257])

$$\forall p \in \mathcal{P}_m : |p|_{\infty, [a,b]} \leq \frac{(m+1)^2}{|b-a|} \int_a^b |p(t)| dt. \quad (3.21)$$

Using (3.21) for $p = p_i^{(k)}$, and for $k \in \{1, \dots, m\}$ we have

$$|p_i^{(k)}|_{\infty, [a,b]} \leq \frac{(m+1-k)^2}{|b-a|} \int_a^b |p_i^{(k)}(t)| dt. \quad (3.22)$$

Since the polynomial $p_i^{(k)}$ has at most $(m-k)$ sign changes in $[a, b]$ we obtain

$$\int_a^b |p_i^{(k)}(t)| dt \leq 2(m+1-k) |p_i^{(k-1)}|_{\infty, [a,b]}. \quad (3.23)$$

Using (3.22), and (3.23) we obtain

$$|p_i^{(k)}|_{\infty, [a,b]} \leq 2 \frac{(m+1-k)^3}{|b-a|} |p_i^{(k)}|_{\infty, [a,b]}. \quad (3.24)$$

By induction on k , we have

$$|p_i^{(k)}|_{\infty, [a,b]} \leq 2^k \frac{(m(m-1) \cdots (m+1-k))^3}{|b-a|^k} |p_i|_{\infty, [a,b]}. \quad (3.25)$$

Let us set $C_m := 2^m (m!)^3 (m+1)^2$. The last inequality and (3.21) imply

$$|p_i^{(k)}|_{\infty, [a,b]} \leq \frac{C_m}{|b-a|^{k+1}} \int_a^b |p_i(t)| dt.$$

Thus

$$\Delta_i \leq C_m \int_a^b |p_i(t)| dt \sum_{k=0}^m \frac{|U_{b,i}^{(m-k)}(b) - U_{a,i}^{(m-k)}(b)|}{|b-a|^{k+1}}.$$

Now Whitney's condition (1.1) implies

$$\Delta_i \leq \frac{C_m}{|b-a|} \omega(|b-a|) \int_a^b |p_i(t)| dt.$$

Therefore

$$\begin{aligned} \ll P; U_b - U_a \gg &\leq \frac{C_m}{|b-a|} \omega(|b-a|) \int_a^b \sum_{i=1}^n |p_i(t)| dt \\ &\leq \frac{\sqrt{n} C_m}{|b-a|} \omega(|b-a|) \int_a^b \|P(t)\| dt, \end{aligned}$$

and

$$K_{a,b} \leq \sqrt{n} C_m \omega(|b-a|),$$

as desired. \square

Now we will prove the stability properties of the extremal extension.

Proposition 3.10. *Let U be a m -jet. Suppose that there exists a concave modulus of continuity denoted by ω such that*

$$\forall a \neq b \in \text{dom}(U), K_{a,b} \leq \omega(|b-a|). \quad (3.26)$$

Then the extremal extension u of U satisfies

$$\forall x, y \in \mathbb{K}, \|u^{(m)}(x) - u^{(m)}(y)\| \leq 3 \omega\left(\frac{|y-x|}{3}\right). \quad (3.27)$$

In other words, the associated extension scheme is Ω -stable. Furthermore the uniqueness of the extremal extension implies that this scheme is a self-reproducing scheme.

For the definitions of Ω -stability, and self-reproducing of this scheme see [24].

Proof. Denote by A the domain of U . Let u the extremal extension of U . Let $x, y \in \mathbb{R}$ with $x < y$. We consider two cases.

Case 1. There exist $a, b \in A$ such that $a \leq x < y \leq b$, $]a, b[\cap A = \emptyset$.

Using the definition of u in restriction to $[a, b]$ we have

$$u^{(m)}(y) - u^{(m)}(x) = \frac{K_{a,b}}{b-a} \int_x^y \frac{L_{a,b}(t)}{\|L_{a,b}(t)\|} dt.$$

It follows that

$$\|u^{(m)}(y) - u^{(m)}(x)\| \leq \frac{y-x}{b-a} K_{a,b} \leq \frac{y-x}{b-a} \omega(b-a).$$

Since ω is concave we have

$$\|u^{(m)}(y) - u^{(m)}(x)\| \leq \omega(y-x).$$

Case 2. There exist $a, b, c, d \in A$ such that $a \leq x < b \leq c \leq y \leq d$, with $]a, b[\cap A = \emptyset$, and $]c, d[\cap A = \emptyset$.

Using the definitions of u in restriction to $[a, b]$, $[b, c]$, and $[c, d]$ we have

$$\begin{aligned} u^{(m)}(y) - u^{(m)}(x) &= \frac{K_{c,d}}{c-d} \int_c^y \frac{L_{c,d}(t)}{\|L_{c,d}(t)\|} dt + \frac{K_{b,c}}{c-b} \int_b^c \frac{L_{b,c}(t)}{\|L_{b,c}(t)\|} dt \\ &\quad + \frac{K_{a,b}}{b-a} \int_x^b \frac{L_{a,b}(t)}{\|L_{a,b}(t)\|} dt. \end{aligned}$$

We obtain the following inequalities

$$\|u^{(m)}(y) - u^{(m)}(x)\| \leq \frac{y-c}{d-c} K_{c,d} + \frac{c-b}{c-b} K_{c,b} + \frac{b-x}{b-a} K_{a,b}.$$

Since ω is concave we have

$$\|u^{(m)}(y) - u^{(m)}(x)\| \leq 3 \omega\left(\frac{y-x}{3}\right).$$

□

4 Annex

Recall that \mathcal{A} is a d -dimensional affine space, with $d := (m+1)n$.

Lemma 4.1. *Let us consider the hyperplane*

$$H := \{\Lambda(P) \in \mathcal{P} : \langle O\Lambda(P); V \rangle = \alpha\},$$

with $V \in R^d \setminus \{0\}$, and $\alpha \in \mathbb{R}$.

Let $a, b \in \mathbb{R}$ with $a \neq b$. Then there exists a function $\phi \in \mathbb{L}_\infty([a, b], \mathbb{R}^d)$ such that

$$H := \{\Lambda(P) : \int_a^b \langle P(t); \phi(t) \rangle dt = \alpha, P \in \mathcal{P}\}.$$

In other words, the hyperplane H can be represented in the last form.

Proof. Denote by $(v_{i,k})_{i=1,\dots,n,k=0,\dots,m}$ the coordinates of V . For $j \in \{0, \dots, m\}$, we set $W_j := \text{Span}(\{1, t, \dots, t^{j-1}, t^{j+1}, \dots, t^m\})$. There exists $\psi_j \in \mathcal{P}_m$ such that

$$\begin{cases} \int_a^b \psi_j(t) h(t) dt = 0, \forall h \in W_j \\ \int_a^b \psi_j(t) t^j dt = 1. \end{cases}$$

Denote by ψ_j the polynomial which satisfies the last equalities.

Let us set $\phi := (\phi_1, \dots, \phi_n)$ with $\phi_i := \sum_{k=0}^m v_{i,k} \psi_k$.

It is easy to verify that ϕ is a good candidate. Let $P \in \mathcal{P}$ we have

$$\begin{aligned} \int_a^b \langle P(t); \phi(t) \rangle dt &= \int_a^b \sum_{i=1}^n p_i(t) \phi_i(t) dt \\ &= \sum_{i=1}^n \int_a^b p_i(t) \phi_i(t) dt \\ &= \sum_{i=1}^n \int_a^b (\sum_{k=0}^m p_{i,k} t^k) (\sum_{j=0}^m v_{i,j} \psi_j(t)) dt \\ &= \sum_{i=1}^n \sum_{j=0}^m \sum_{k=0}^m (p_{i,k} v_{i,j} \int_a^b t^k \psi_j(t) dt) \\ &= \sum_{i=1}^n \sum_{j=0}^m p_{i,j} v_{i,j} \\ &= \langle O\Lambda(P); V \rangle. \end{aligned}$$

□

Let us recall a Taylorian formula associated to bipoctual m -fields.

Lemma 4.2. Let $a \neq b \in \mathbb{R}$. Let $\phi \in \mathcal{C}_\infty^{m+1}([a, b]; \mathbb{R}^n)$. Denote by $U_{a,m}$ (resp. $U_{b,m}$) the m -Taylorian expansion of ϕ at a (resp. b).

$$U_{a,m}(x) := \sum_{k=0}^m \frac{(x-a)^k}{k!} \phi^{(k)}(a).$$

Then we have the following Taylorian formula

$$U_{b,m}(x) - U_{a,m}(x) = \int_a^b \frac{(x-t)^m}{m!} \phi^{(m+1)}(t) dt. \quad (4.1)$$

Proof. By induction on m , and by integration by parts over $[a, b]$ it is easy to prove this Lemma. For $m = 0$ we have

$$U_{b,0}(x) - U_{a,0}(x) = \phi(b) - \phi(a) = \int_a^b \phi^{(1)}(t) dt.$$

Suppose that (4.1) is true for $m - 1$ then

$$\begin{aligned}
 \int_a^b \frac{(x-t)^m}{m!} \phi^{(m+1)}(t) dt &= \frac{(x-b)^m}{(m)!} \phi^{(m)}(b) - \frac{(x-a)^m}{(m)!} \phi^{(m)}(a) \\
 &+ \int_a^b \frac{(x-t)^{m-1}}{(m-1)!} \phi^{(m)}(t) dt \\
 &= \frac{(x-b)^m}{(m)!} \phi^{(m)}(b) - \frac{(x-a)^m}{(m)!} \phi^{(m)}(a) \\
 &+ U_{b,m-1}(x) - U_{a,m-1}(x) \\
 &= U_{b,m}(x) - U_{a,m}(x).
 \end{aligned}$$

□

Noting that more generally, for $k \in \{0, \dots, m\}$ we have the following formula

$$U_{b,m}^{(k)}(x) - U_{a,m}^{(k)}(x) = \int_a^b \frac{(x-t)^{m-k}}{(m-k)!} \phi^{(m+1)}(t) dt. \quad (4.2)$$

4.1 Basic properties of $\ll ; \gg$

Let $P, Q \in \mathcal{P}$. Recall that (p_1, \dots, p_n) (resp. (q_1, \dots, q_n)) are the coordinates of P , and Q , and that $\ll ; \gg$ is defined by the following formula

$$\ll Q; P \gg := \sum_{i=1}^n \sum_{k=0}^m ((-1)^{m-k} q_i^{(k)}(b) p_i^{(m-k)}(b)).$$

For any $P \in \mathcal{P}$, the map $\ll ; P \gg : Q \in \mathcal{P} \longrightarrow \ll Q; P \gg \in \mathbb{R}$ is a linear function defined on \mathcal{P} .

Let us consider the map Ψ that sends P to $\ll ; P \gg$. Then Φ is linear and invertible.

Let $P \in \text{null } \Phi$. For $i \in \{1, \dots, n\}$, and for $j \in \{0, \dots, m\}$ let us consider the following polynomials $Q_{i,j} := t^j \mathbf{e}_i$. We have

$$\ll Q_{i,j}; P \gg = \sum_{k=0}^m ((-1)^{m-k} (t^j)^{(k)}(b) p_i^{(m-k)}(b)) = (-1)^{m-j} j! p_i^{(m-j)}(b) = 0.$$

Thus $p_i^{(k)}(b) = 0$ for every $k \in \{0, \dots, m\}$ hence $p_i = 0$.

In other words, $\text{null } \Phi = \{0\}$ and we can identify any linear function of the dual of \mathcal{P} to the map $\ll ; P \gg$, with $P \in \mathcal{P}$.

Lemma 4.3. *Let $U = \{U_a, U_b\}$ be a bimonotone m -jet. Let $g \in \mathcal{E}(U)$. Then*

$$\ll Q, U_{a,b} \gg = \int_a^b \langle Q(t); g^{(m+1)}(t) \rangle dt, \quad \forall Q \in \mathcal{P}. \quad (4.3)$$

Proof. Let us set $U_{a,b} = U_b - U_a$. Let $Q \in \mathcal{P}$. Using the Taylorian formula (4.2) we have

$$U_{a,b}^{(r)}(x) = \int_a^b \frac{(x-t)^{m-r}}{(m-r)!} g^{(m+1)}(t) dt, \quad x \in \mathbb{R}, \quad r \in \{0..m\}. \quad (4.4)$$

For $k \in \{1, \dots, n\}$, let us consider $\Delta_k := \sum_{j=0}^m (-1)^j q_k^{(j)}(b) U_{a,b,k}^{(m-j)}(b)$. Using (4.4) for $r = m - j$ and $x = b$, we have

$$\begin{aligned} \Delta_k &= \sum_{j=0}^m (-1)^j q_k^{(j)}(b) \int_a^b \frac{(x-t)^j}{(j)!} g^{(m+1)}(t) dt \\ &= \int_a^b \sum_{j=0}^m \frac{(b-t)^j}{j!} q_k^{(j)}(b) g^{(m+1)}(t) dt. \end{aligned} \quad (4.5)$$

The classical m -Taylorian expansion is exact for polynomials of degree at most m therefore

$$q_k(t) = \sum_{j=0}^m (-1)^j \frac{(b-t)^j}{(m-k)!} q_k^{(j)}(b).$$

Thus

$$\Delta_k = \int_a^b q_k(t) g_k^{(m+1)}(t) dt,$$

hence

$$\ll Q, U_{a,b} \gg = \sum_{k=1}^n \Delta_k = \int_a^b \langle Q(t); g^{(m+1)}(t) \rangle dt,$$

and the equality (4.3) is proved. \square

4.2 Uniqueness of supporting hyperplane with one more supplementary hypothesis

Under the following additional hypothesis

$$\|\phi(t)\| = 1, \quad \forall t \in [a, b], \quad (4.6)$$

we give a new proof of the uniqueness of the supporting hyperplane containing $\Lambda(L)$. We will use the same notations as in Lemma 3.1. We will develop this proof for its neatness.

Suppose (4.6), and let us consider the following sets

$$I_+ := \{t \in [a, b] : \langle L(t); \delta(t) \rangle > 0\}, \quad I_- := \{t \in [a, b] : \langle L(t); \delta(t) \rangle < 0\},$$

and

$$I_0 := \{t \in [a, b] : \langle L(t); \delta(t) \rangle = 0\}.$$

We will prove that I_+ and I_- are null set. Since $\Lambda(L) \in H_\phi \cap \mathcal{B}$ we have the two following equalities

$$\int_a^b \langle L(t); \delta(t) \rangle dt = 0, \quad (4.7)$$

and

$$\int_a^b \langle L(t); \phi(t) \rangle dt = \int_a^b \|L(t)\| dt = b - a.$$

The second equality implies

$$\int_{t \in I_+} \|L(t)\| \|\phi(t)\| dt + \int_{t \in I_-} \langle L(t); \phi(t) \rangle dt \geq \int_a^b \|L(t)\| dt,$$

and by using (4.6) we have

$$\int_{t \in I_+} \|L(t)\| dt + \int_{t \in I_-} \langle L(t); \phi(t) \rangle dt \geq \int_a^b \|L(t)\| dt.$$

Since $\phi = \frac{L}{\|L\|} + \delta$ we obtain :

$$\int_{t \in I_-} \langle L(t); \delta(t) \rangle dt \geq 0.$$

Thus I_- is a null set. By (4.7,) this result implies that I_+ is also a null set. Hence

$$\int_a^b \left\langle \frac{L(t)}{\|L(t)\|}; \delta(t) \right\rangle dt = 0. \quad (4.8)$$

Squaring and integrating over $[a, b]$ the relation (4.6), we obtain

$$\int_a^b \left\| \frac{L(t)}{\|L(t)\|} + \delta(t) \right\|^2 dt \leq b - a. \quad (4.9)$$

By (4.8), and the expansion of the last equality we have

$$\int_a^b \|\delta(t)\|^2 dt \leq 0.$$

Therefore δ is a null function.

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